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ScienceDirect
Journal of Approximation Theory 159 (2009) 39–60
Journal of Approximation Theory
www.elsevier.com/locate/jat

Multi-segmental representations and approximation of set-valued functions with 1D images

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Received 21 July 2008; accepted 13 November 2008
Available online 30 November 2008

Communicated by C.K. Chui and H.N. Mhaskar

Dedicated to the memory of G.G. Lorentz

Abstract

In this work univariate set-valued functions (SVFs, multifunctions) with $1D$ compact sets as images are considered. For such a continuous SFV of bounded variation (CBV multifunction), we show that the boundaries of its graph are continuous, and inherit the continuity properties of the SVF. Based on these results we introduce a special class of representations of CBV multifunctions with a finite number of 'holes' in their graphs. Each such representation is a finite union of SVFs with compact convex images having boundaries with continuity properties as those of the represented SVF. With the help of these representations, positive linear operators are adapted to SVFs. For specific positive approximation operators error estimates are obtained in terms of the continuity properties of the approximated multifunction.

Keywords: Compact sets; Minkowski sum; Segment functions; Set-valued functions; Multi-segmental representation; Selection; Positive linear approximation operators; Continuous set-valued functions of bounded variation; Error estimates

1. Introduction

The classical adaptation of linear positive approximation operators for univariate real-valued functions, to univariate set-valued functions (SVFs, multifunctions), is by replacing sums between numbers by Minkowski sums of sets (see e.g. [1–3]). This adaptation is capable of approximating SVFs with compact convex images only [3]. Other adaptations effective for SVFs with general compact images are reviewed in [4].

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2008.11.013

Here we study univariate SVFs with compact images in $\mathbb{R}$. This case is easier to analyze, and can be considered as an important first step towards future work on SVFs with compact images in $\mathbb{R}^n$.

We limit our investigation to continuous SVFs of bounded variation (CBV multifunctions), since as we show, the boundary of the graph of such a multifunction is a collection of real continuous functions, with continuity properties inherited from those of the SVF. This leads to the observation that in the class $\mathcal{F}$ of CBV multifunctions with a finite number of 'holes' in their graphs, any SVF can be represented (in many ways) as a union of a finite number of segment functions, intersecting only at their boundaries, (segment functions are SVFs with compact convex images in $\mathbb{R}$). We call such a representation 'multi-segmental' (MSR), and prove the existence of a specific MSR, termed 'topological', such that the boundaries of the segment functions inherit the continuity properties of the SVF. Furthermore, we derive conditions on a multifunction in $\mathcal{F}$, guaranteeing the uniqueness of a topological MSR with minimal number of segment functions.

Given a topological MSR of a multifunction $F \in \mathcal{F}$, we define a positive linear operator on F, as the union of the SVFs, obtained by the application of the classical adaptation of the operator to each segment function in the MSR. By this approach, the application of a positive linear operator to a multifunction in $\mathcal{F}$ reduces to its application to the boundaries of a topological MSR.

For operators, which approximate continuous real-valued functions, our approach provides error estimates, in terms of the regularity properties of the approximated SVF. For the classical Bernstein polynomial operators and the Schoenberg spline operators, explicit error bounds are given. These approximation results are illustrated by an example.

The paper is organized as follows. The next section contains notation and basic definitions. In Section 3 certain properties of the graphs of continuous SVFs with $1D$ images are discussed. Theorems on the continuity and the properties of the boundaries of the graph of a CBV multifunction are stated. Most proofs are deferred to Appendices A and B. Section 4 introduces the notion of multi-segmental representations of SVFs in $\mathcal{F}$, and presents results on their boundaries. In Section 5 we define a topological MSR and prove its existence in a constructive way. Section 5.2 introduces a specific class of topological MSRs, and presents conditions on the graph of a multifunction in $\mathcal{F}$, guaranteeing its uniqueness. Section 6 is devoted to positive linear operators defined by means of topological MSRs, and to their approximation properties. The approximation results are specialized in Section 7 to two well-known positive operators.

2. Preliminaries

In this section we present basic definitions and notation. By $K(\mathbb{R})$ we denote the collection of all compact non-empty subsets of $\mathbb{R}$, by $Co(\mathbb{R})$ — the collection of all convex sets in $K(\mathbb{R})$ (closed intervals). For a set $A \in K(\mathbb{R})$ its convex hull is denoted by $\mathrm{co}(A)$, its closure by $\mathrm{cl}(A)$, its interior by $\mathrm{int}(A)$, its boundary by ∂A and its Lebesgue measure by $\mu(A)$. The closed interval in $\mathbb{R}$, $[a-r, a+r]$, $r > 0$ is denoted by $\mathcal{B}(a, r)$. A line segment between $p, q \in \mathbb{R}^n$, including the endpoints, is $[p, q]$. We also use the standard notation $[p, q)$, $(p, q]$ and (p, q) where one or both points are not included. $\mathcal{C}([a, b])$ is the collection of all continuous real-valued functions on $[a, b]$, and π_m is the space of univariate polynomials of degree $\leq m$.

A linear Minkowski combination of two sets A and B is

$$\lambda A + \mu B = \{\lambda a + \mu b : a \in A, b \in B\},$$

with $\lambda, \mu \in \mathbb{R}$.

The Euclidean distance from a point $a \in \mathbb{R}$ to a set $B \in K(\mathbb{R})$ is defined as

$$\operatorname{dist}(a, B) = \inf_{b \in B} |a - b|.$$

The Hausdorff distance between two sets $A, B \in K(\mathbb{R})$ is defined by

$$\operatorname{haus}(A, B) = \max\left\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\right\}.$$

Obviously, if $A = [a_1, a_2]$, $B = [b_1, b_2]$, then

$$\operatorname{haus}(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}. \tag{1}$$

The set of projections of $a \in \mathbb{R}$ into a set $B \in K(\mathbb{R})$ is

$$\Pi_B(a) = \{b \in B : |a - b| = \operatorname{dist}(a, B)\}.$$

In this paper we investigate SVFs $F : [a, b] \to K(\mathbb{R})$, which are continuous in the Hausdorff metric. We represent them as a union of **segment functions**, namely SVFs with images in $Co(\mathbb{R})$. For a multifunction F, any single-valued function $f : [a, b] \to \mathbb{R}$ with $f(x) \in F(x), \forall x \in [a, b]$ is called a **selection** of F.

We recall the notion of the modulus of continuity of a function f defined on $[a, b]$ with images in a metric space (X, ρ)

$$\omega_{[a,b]}(f, \delta) = \sup\{\rho(f(x), f(y)) : |x - y| \le \delta, x, y \in [a, b]\}, \quad \delta \in \mathbb{R}_+. \tag{2}$$

Here X is either $K(\mathbb{R})$ or $\mathbb{R}$, and ρ is either the Hausdorff metric or the standard metric in $\mathbb{R}$ respectively.

Among the continuous functions we denote by $\operatorname{Lip}([a, b], L)$ the collection of all Lipschitz continuous functions, namely functions satisfying

$$\rho(f(x), f(y)) \le L|x - y|, \quad \forall x, y \in [a, b].$$

We also recall the definition of the total variation of $f : [a, b] \to X$. The total variation of f on $[a, b]$ is defined by

$$V_a^b(f) = \sup_{\chi} V(f, \chi),$$

where $V(f, \chi) = \sum_{i=1}^{N} \rho(f(x_i), f(x_{i-1}))$ is the variation of f on a partition $\chi = \{x_0, \ldots, x_N\}$, $a \le x_0 < x_1 < \cdots < x_N \le b$.

A function f is of bounded variation if $V_a^b(f) < \infty$. For f of bounded variation, we consider the non-decreasing function

$$v_f(x) = V_a^x(f), \quad x \in [a, b]. \tag{3}$$

It can be shown (see e.g. [5]), that f is continuous iff v_f is continuous. Moreover,

$$\omega_{[a,b]}(f, \delta) \le \omega_{[a,b]}(v_f, \delta). \tag{4}$$

We denote the set of all functions $f : [a, b] \to X$ which are continuous and of bounded variation by $CBV([a, b])$.

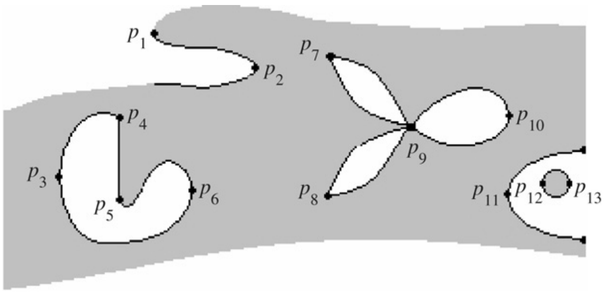

Fig. 3.1. Grey — Graph(F), black — boundaries of the holes of F.

3. Continuous SVFs with images in $\mathbb{R}$

Our aim is to represent set-valued functions as a union of segment functions. First we introduce some notions and notation central to our analysis.

For a set-valued function $F : [a, b] \to K(\mathbb{R})$ the graph is the $2D$ set

$$\mathrm{Graph}(F) = \{(x, y) : x \in [a, b], y \in F(x)\}.$$

Note that the graph of a continuous SVF is closed. Consider the set

$$\mathrm{Graph}(\mathrm{co}\, F) \setminus \mathrm{Graph}(F), \tag{5}$$

where $\mathrm{co}\, F : [a, b] \to Co(\mathbb{R})$ is defined by $(\mathrm{co}\, F)(x) = \mathrm{co}\,(F(x))$, $x \in [a, b]$.

We call a maximal connected open subset H of (5) a **hole of *F***. The collection of all such holes is denoted by $\mathcal{H}(F)$. The number of holes in $\mathcal{H}(F)$ is denoted by $|\mathcal{H}(F)|$.

The **boundary** of a hole $H \in \mathcal{H}(F)$ in the graph of F is

$$\widetilde{\partial} H = cl(H) \cap \mathrm{Graph}(F).$$

An **interior hole** of F is a hole H for which $cl(H) \setminus H = \widetilde{\partial} H$. All other holes are termed **boundary holes**. In Fig. 3.1 the four holes with boundaries containing the points $p_3, \ldots, p_{10}$ are interior holes, the other two are boundary holes.

The **width** of a hole H is denoted by $\Delta_H = [x_H^l, x_H^r]$ with

$$x_H^l = \inf\{x : \exists y \text{such that} (x, y) \in H\},$$
$$x_H^r = \sup\{x : \exists y \text{such that} (x, y) \in H\}.$$

We call the set $H(x) = \{y : (x, y) \in H\}$ a **cross-section** of H at x.

In the following we define points on Graph(F), where locally the topology of the images of a multifunction F changes. These points play a central role in our analysis.

Definition 3.1. A point $(x, y) \in \mathrm{Graph}(F)$ is called a **point of change of topology of *F*** (PCT) if for any $\varepsilon > 0$ small enough there exists $\delta(\varepsilon) \in R$, such that $\forall z$ satisfying

$$\min\{0, \delta(\varepsilon)\} < z - x < \max\{0, \delta(\varepsilon)\}, \tag{6}$$

$F(x) \cap \mathcal{B}(y, \varepsilon)$ and $F(z) \cap \mathcal{B}(y, \varepsilon)$ consist of a different number of intervals (a single point is considered as an interval of zero length).

It is easy to see, that any point of change of topology p is associated with some hole $H \in \mathcal{H}(F)$, namely $p \in \widetilde{\partial} H$. In Fig. 3.1 the points p_1, p_2, p_3, all the points on $(p_4, p_5]$ and the points $p_6, \dots, p_{13}$ are PCTs of F.

Among the points of change of topology we mark a special type of PCTs. The absence of points of this type is necessary for the continuity of SVFs.

Definition 3.2. A point of change of topology (x, y) is called **singular** if $F(z) \cap \mathcal{B}(y, \varepsilon) = \emptyset$, with ε and z as in Definition 3.1.

In Fig. 3.1 p_1, p_{12}, p_{13} and the points of $(p_4, p_5]$ are singular PCTs.

Remark 3.3. It follows directly from Definition 3.2 that

(i) if for some $x \in \Delta_H$, $\widetilde{\partial} H \cap \{(x, y) : y \in F(x)\}$ contains a vertical segment, then the interior points of this segment are singular.
(ii) any isolated point of $\mathrm{Graph}(F)$ is a singular PCT.

Lemma 3.4. *A continuous multifunction F does not have singular PCTs.*

Proof. Suppose (x, y), $y \in F(x)$ is a singular PCT. By Definition 3.2 there exist $\varepsilon > 0$ and $\delta(\varepsilon)$ such that $\forall z$ satisfying (6), $\mathrm{dist}\,(y, F(z)) > \varepsilon$. Then $\mathrm{haus}(F(x), F(z)) > \varepsilon$, implying that F is discontinuous at x. $\square$

We define the **lower boundary** and the **upper boundary** of $F : [a, b] \to K(\mathbb{R})$ as

$$f^{\mathrm{low}}(x) = \min\{y : y \in F(x)\}, \qquad f^{\mathrm{up}}(x) = \max\{y : y \in F(x)\}, \quad x \in [a, b]. \tag{7}$$

We show that for a continuous multifunction F, f^{low} and f^{up} are continuous.

Theorem 3.5. *Let $F : [a, b] \to K(\mathbb{R})$ be a continuous set-valued function. Then f^{low} and f^{up} are continuous, and satisfy*

$$\max(\omega_{[a,b]}(f^{\mathrm{low}}, \delta), \omega_{[a,b]}(f^{\mathrm{up}}, \delta)) \le \omega_{[a,b]}(F, \delta).$$

Proof. In the following f is either f^{low} or f^{up}. Consider $|f(x) - f(z)|$ with $|x - z| \le \delta$, $\delta > 0$. Note that by (7), $|f(x) - f(z)|$ is either $\mathrm{dist}(f(x), F(z))$ or $\mathrm{dist}(f(z), F(x))$ or both. Then

$$|f(x) - f(z)| \le \mathrm{haus}(F(x), F(z)) \le \omega_{[a,b]}(F, |x - z|),$$

which implies the claim of the theorem. $\square$

For any hole $H \in \mathcal{H}(F)$ we define its lower and upper boundaries as

$$\begin{aligned} b_H^{\mathrm{low}}(x) &= \inf\{y : (x, y) \in H\}, \quad x \in (x_H^l, x_H^r) \\ b_H^{\mathrm{up}}(x) &= \sup\{y : (x, y) \in H\}, \quad x \in (x_H^l, x_H^r). \end{aligned} \tag{8}$$

These functions and f^{low}, f^{up}, are termed the boundaries of F, and are denoted by

$$\partial F = \{b_H^{\mathrm{low}}, b_H^{\mathrm{up}} : H \in \mathcal{H}(F)\} \bigcup \{f^{\mathrm{low}}, f^{\mathrm{up}}\}. \tag{9}$$

In the next two theorems we show that in addition to the claim of Theorem 3.5, all the boundaries of $F \in CBV([a, b])$ are continuous, with regularity properties determined by those of F.

Theorem 3.6. *For $F \in CBV([a,b])$ and $H \in \mathcal{H}(F)$, the functions $b_H^{\text{low}}(x)$, $b_H^{\text{up}}(x)$ are continuous on Δ_H and coincide at x_H^l (x_H^r) whenever $a < x_H^l$ ($x_H^r < b$).*

The proof of the theorem is a direct conclusion of a series of lemmas, stated and proved in Appendix A.

It follows from Theorem 3.6, that $H \in \mathcal{H}(F)$ is determined by $b_H^{\text{low}}(x)$ and $b_H^{\text{up}}(x)$.

Theorem 3.7. *Let $F \in CBV([a,b])$. Then any $f \in \partial F$ satisfies*

$$\omega_{[a,b]}(f,\delta) \le \omega_{[a,b]}(v_F,\delta), \quad \delta > 0.$$

Moreover if $F \in \mathrm{Lip}([a,b],L)$ then $f \in \mathrm{Lip}([a,b],L)$.

The proof of this theorem is deferred to Appendix B.

From now on we consider SVFs in the class $\mathcal{F}([a,b])$ of continuous multifunctions of bounded variation with a finite number of holes. The last assumption facilitates the derivation of multi-segmental representations.

4. Multi-segmental representations of SVFs

In this section we show that the graph of any $F \in \mathcal{F}([a,b])$ can be represented as a union of the graphs of a finite number of continuous convex-valued functions, with graphs intersecting at most at their boundaries. In the next section we propose a special construction of such a representation of F with at most $|\mathcal{H}(F)| + 1$ segment functions, having boundaries with continuity properties determined by those of F.

First we notice that any image $F(x)$ of $F \in \mathcal{F}([a,b])$ for $x \in [a,b]$ consists of a finite number of disjoint, closed segments of R. Namely,

$$F(x) = \bigcup_{n=1}^{N(x)} I_n(x), \quad x \in [a,b] \tag{10}$$

where $\{I_n(x)\}$ are closed intervals such that $y < g$ for any $y \in I_n(x)$ and $g \in I_{n+1}(x)$, $n = 1,\ldots,N(x)-1$. Clearly, the endpoints of $I_n(x)$ correspond to boundary points of Graph(F).

If $N(x) \equiv 1$ then F is a segment function and can be represented as

$$F(x) = [f^{\text{low}}(x), f^{\text{up}}(x)], \quad f^{\text{low}}(x) \le f^{\text{up}}(x), \quad x \in [a,b]. \tag{11}$$

Obviously $F(x) = \mathrm{co}(F(x))$. It is easy to prove that the segment function F is continuous iff f^{low} and f^{up} are continuous.

Definition 4.1. A multifunction $F \in \mathcal{F}([a,b])$ has a **multi-segmental representation** (MS-representation, MSR), if there is a natural number N such that

$$F(x) = \bigcup_{n=1}^{N} F_n(x) = \bigcup_{n=1}^{N} [f_n^{\text{low}}(x), f_n^{\text{up}}(x)], \quad x \in [a,b], \tag{12}$$

where $F_n = [f_n^{\text{low}}, f_n^{\text{up}}]$, $n = 1,\ldots,N$ are segment functions, and where for $x \in [a,b]$

$$f_1^{\text{low}}(x) \le f_1^{\text{up}}(x) \le f_2^{\text{low}}(x) \le \cdots \le f_{N-1}^{\text{up}}(x) \le f_N^{\text{low}}(x) \le f_N^{\text{up}}(x). \tag{13}$$

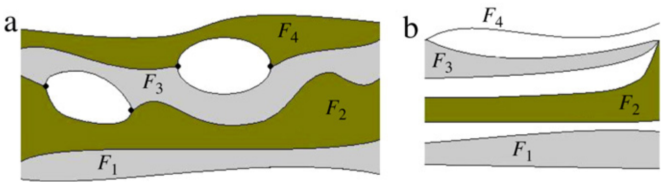

Fig. 4.1. (a) The graph of a multi-segmental representation. (b) The graph of a SVF with its natural MS-representation.

We denote such a multi-segmental representation by $\mathcal{R} = \{F_n, n = 1, \ldots, N\}$. $\mathcal{R}$ is determined by the boundary selections

$$\mathcal{B}(\mathcal{R}, F) = \{f_n^{\text{low}}(x),\ f_n^{\text{up}}(x),\ n = 1, \ldots, N\}. \tag{14}$$

Each selection in (14) is called a **MS-boundary**. An example of such a representation is shown in Fig. 4.1(a). In general, the MS-boundaries may be quite arbitrary. Yet there is a class of SVFs with a MS-representation determined by their boundaries.

Definition 4.2. A MS-representation $\mathcal{R}$ of a multifunction $F \in \mathcal{F}([a, b])$ is called **natural** if $\mathcal{B}(\mathcal{R}, F) = \partial F$ and for any $1 \le i \le N - 1, \{x : F_i(x) \bigcap F_{i+1}(x) \neq \emptyset\} \subset \{a, b\}$. The boundaries $\mathcal{B}(\mathcal{R}, F)$ are called natural MS-boundaries.

Clearly, the natural MS-representation is unique. The graph of a multifunction F with its natural MSR is shown in Fig. 4.1(b).

Remark 4.3. Any $F \in \mathcal{F}([a, b])$ determines a **natural partition** of $[a, b]$, $\chi_F = \{x_0, \ldots, x_M\}$, consisting of the distinct points among $\{a, b, x_H^l, x_H^r : H \in \mathcal{H}(F)\}$. On each $\Delta_i = [x_i, x_{i+1}], i = 0, \ldots, M-1, \partial F|_{\Delta_i}$ consists of $f^{\text{up}}|_{\Delta_i}, f^{\text{low}}|_{\Delta_i}, b_H^{\text{up}}|_{\Delta_i}, b_H^{\text{low}}|_{\Delta_i}$ for $H \in \mathcal{H}(F)$ such that $\Delta_H \supset \Delta_i$.

It is easy to see that

$$F(x) = [f^{\text{low}}(x), f^{\text{up}}(x)] \setminus \bigcup \{(b_H^{\text{low}}(x), b_H^{\text{up}}(x)) : H \in \mathcal{H}(F), \Delta_i \subset \Delta_H\}, \quad x \in \Delta_i$$

which is equivalent to

$$F|_{\Delta_i}(x) = \bigcup_{n=1}^{N_i} F_n^i(x) = \bigcup_{n=1}^{N_i} [(f^i)_n^{\text{low}}(x), (f^i)_n^{\text{up}}(x)], \quad x \in \Delta_i, \tag{15}$$

with $(f^i)_n^{\text{low}}(x) \le (f^i)_n^{\text{up}}(x) \le (f^i)_{n+1}^{\text{low}}(x)$, $x \in \Delta_i$. Thus any $F \in \mathcal{F}([a, b])$ has a unique **piecewise natural** MS-representation.

To illustrate Remark 4.3, consider $F \in \mathcal{F}([a, b])$ with graph as shown in Fig. 5.1. For this F the natural partition is $\chi_F = \{a, x_H^l, x_H^r, b\}$, $\Delta_0 = [a, x_H^l]$, $\Delta_1 = [x_H^l, x_H^r]$, $\Delta_2 = [x_H^r, b]$, and (15) holds with $N_0 = 2$, $N_1 = 2$, $N_2 = 1$. The natural MS-boundaries on Δ_0 are

$$(f^0)_1^{\text{low}} = f^{\text{low}}|_{\Delta_0}, \qquad (f^0)_1^{\text{up}} = b_{\tilde{H}}^{\text{low}}, \qquad (f^0)_2^{\text{low}} = b_{\tilde{H}}^{\text{up}}, \qquad (f^0)_2^{\text{up}} = f^{\text{up}}|_{\Delta_0}.$$

Similarly, natural MS-boundaries on Δ_1 and Δ_2 can be easily defined from Fig. 5.1.

From now on we consider only multi-segmental representations with continuous MS-boundaries and call them **continuous MS-representations**. Note that in this case the segment functions are continuous, which is not the case in Fig. 4.2.

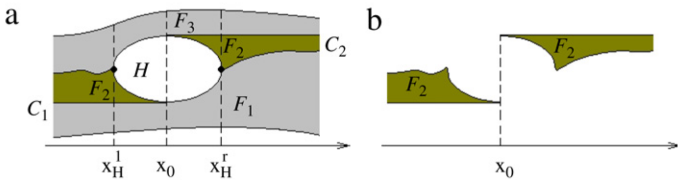

Fig. 4.2. (a) The graph of a discontinuous MS-representation. (b) The graph of the discontinuous segment function F_2.

It is clear that for any MS-representation $\mathcal{R}$ of F, $\bigcup_{f\in\partial F}\text{Graph}(f) \subset \bigcup_{b\in\mathcal{B}(\mathcal{R},F)}\text{Graph}(b)$, since $\mathcal{R}$ consists of segment functions only. Yet for continuous MSRs a stronger result holds.

Lemma 4.4. *Let $F \in \mathcal{F}([a,b])$ and let $\mathcal{R}$ be a continuous MSR of F. Then for any $f \in \partial F$ there exists $b \in \mathcal{B}(\mathcal{R}, F)$ such that $f(x) = b(x)$, for any x in the domain of the function f.*

Proof. If $f = f^{\text{low}}$ or $f = f^{\text{up}}$, then the claim of the lemma is trivial. Assume that $f = b_H^{\text{up}}$ for some $H \in \mathcal{H}(F)$. First we prove that any point of Graph(f) belongs to the graph of some MS-boundary. Let $y = f(x)$, $x \in \Delta_H$. Obviously $(x, y) \in \text{Graph}(F_i)$, where F_i is some segment function in (12). It is clear that $(x, y) \in \partial\text{Graph}(F_i)$, since $(x, y) \in \partial\text{Graph}(F)$ and $\text{Graph}(F_i) \subset \text{Graph}(F)$. Denote by s_x a selection from $\mathcal{B}(\mathcal{R}, F)$, satisfying $s_x(x) = b_H^{\text{up}}(x)$.

Next we show that there must be a MS-boundary, which coincides with b_H^{up} on Δ_H. Let

$$S = \{s : s \in \mathcal{B}(\mathcal{R}, F), s(x) \geq b_H^{\text{up}}(x), x \in \Delta_H\} = \{f_i^{\text{low}}, f_i^{\text{up}} : n_0 \leq i \leq N\},$$

and let $s_{\min} \in S$ be such that $s_{\min}(x) \leq s(x)$, $x \in [a, b]$ for any $s \in S$. Such a selection exists by (13).

To see that b_H^{up} coincides on Δ_H with $s_{\min}$ assume to the contrary that $s_{\min}(x) > b_H^{\text{up}}(x)$ on a non-empty subset $\widetilde{\Delta}_H$ of Δ_H. By the continuity of $s_{\min}$ and b_H^{up} one may assume that $\widetilde{\Delta}_H$ contains an interior point $x^* \in (x_H^l, x_H^r)$. Thus there exists $s_{x^*} \in \mathcal{B}(\mathcal{R}, F)$ such that $s_{x^*}(x^*) = b_H^{\text{up}}(x^*)$. It follows from Lemma A.4 and the continuity of s_{x^*} and b_H^{up} that $s_{x^*} \in S$. Then

$$b_H^{\text{up}}(x^*) < s_{\min}(x^*) \leq s_{x^*}(x^*) = b_H^{\text{up}}(x^*),$$

which is a contradiction. Thus $b_H^{\text{up}} = s_{\min}$ on Δ_H. A similar proof applies for $f = b_H^{\text{low}}$. $\square$

Fig. 4.2 demonstrates that the continuity assumption of the MS-boundaries is essential in Lemma 4.4. Note that the boundaries $f_2^{\text{low}}(x)$ and $f_2^{\text{up}}(x)$ of the segment function F_2 in Fig. 4.2 are discontinuous at x_0. Indeed

$$f_2^{\text{low}}(x) = \begin{cases} C_1 & x \in [x_H^l, x_0] \\ b_H^{\text{up}}(x), & x \in (x_0, x_H^r], \end{cases}$$

$$f_2^{\text{up}}(x) = \begin{cases} b_H^{\text{low}}(x), & x \in [x_H^l, x_0] \\ C_2, & x \in (x_0, x_H^r], \end{cases}$$

where C_1 and C_2 are constants. Thus there is no $b \in \mathcal{B}(\mathcal{R}, F)$ which coincides either with $b_H^{\text{up}}(x)$ or with $b_H^{\text{low}}(x)$ on the whole domain Δ_H. Yet each point on the boundary of H belongs to the graph of some $\mathcal{B}(\mathcal{R}, F)$.

It follows from Lemma 4.4 that the boundaries of any continuous MSR, contain a special set of selections termed hereafter significant selections.

Definition 4.5. A selection of $F \in \mathcal{F}([a,b])$ is called a **significant selection** if it is continuous on $[a,b]$ and coincides with some $f \in \partial F$ on the domain of f. We denote by s^{low} (s_H^{low}) a significant selection which coincides with f^{low} (b_H^{low}), and similarly for f^{up} and b_H^{up}.

In Fig. 4.1(a) all selections in $\mathcal{B}(\mathcal{R}, F)$ except $f_1^{\text{up}} = f_2^{\text{low}}$ are significant selections. It should be noted that the number of segment functions in a MSR can be reduced by deleting the non-significant selections from the MS-boundaries. In the next section we construct continuous MS-representations with MS-boundaries consisting only of significant selections.

5. Topological multi-segmental representations

One can construct various multi-segmental representations of F by different selections. Also the number of segment functions in (12) may be arbitrarily large. Lemma 4.4 shows that significant selections must participate in any continuous MSR of $F \in \mathcal{F}([a,b])$.

We propose here a MS-representation with MS-boundaries which are special significant selections. These selections inherit the behavior of the boundaries of F and are termed topological. We call the resulting representation topological MSR.

From this point on we use the notation of Remark 4.3.

Definition 5.1. A significant selection s of $F \in \mathcal{F}([a,b])$ is called a **topological selection** if for each Δ_i, where s does not coincide with $f \in \partial F$, there exist $n \in \{1,\ldots,N_i\}$ and $\lambda_n^i \in [0,1]$ such that

$$s|_{\Delta_i} = \lambda_n^i (f^i)_n^{\text{low}} + (1-\lambda_n^i)(f^i)_n^{\text{up}}.$$

Clearly $f^{\text{up}}, f^{\text{low}} \in \partial F$ are topological selections of F and similarly all natural MS-boundaries. An example of a topological selection is the selection in Fig. 5.1 which coincides with b_H^{up} (b_H^{low}) on $[x_H^l, x_H^r]$.

Definition 5.2. An MS-representation with MS-boundaries which are topological selections is called a **topological MS-representation**(TMSR).

5.1. Existence of a topological MS-representation

Our proof of the existence of a TMSR is constructive. The construction uses only a special subset of topological selections.

Definition 5.3. For each $H \in \mathcal{H}(F)$ we define a **pair of topological selections** $t_H^{\text{up}}, t_H^{\text{low}}$ by

$$\begin{aligned} &t_H^{\text{up}} = b_H^{\text{up}}, \quad t_H^{\text{low}} = b_H^{\text{low}} \quad \text{on } \Delta_H \\ &t_H^{\text{up}} = t_H^{\text{low}} \quad \text{on } [a,b] \setminus \Delta_H. \end{aligned}$$

Note that in general a hole $H \in \mathcal{H}(F)$ may have more than one pair of topological selections (see the discussion in Section 5.2). Fig. 5.1 illustrates a hole H with a unique pair of topological selections.

Our construction of a TMSR is recursive. Each step starts with a union of multifunctions representing F and eliminates (“cuts”) one hole of one such multifunction, replacing it by two SVFs. The “cutting” of the hole is along one of its pairs of topological selections. Thus at the

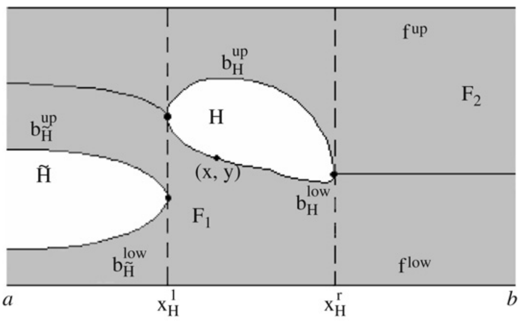

Fig. 5.1.

end of such a step the number of multifunctions representing F is increased by one, while the total number of holes of these SVFs is decreased by one. The number of steps required for the construction of a TMSR of F by this procedure is $|\mathcal{H}(F)|$ which is finite by assumption. We first describe the idea of the construction on an example.

Example 5.4. Consider F with the graph presented in Fig. 5.1.

To construct our TMSR of F, we define a pair of topological selections corresponding to the hole H. The first selection t_H^{low} coincides with b_H^{low} on Δ_1, while on Δ_0 and Δ_2 it is defined as a fixed convex combination of the relevant natural MS-boundaries. More precisely

$$t_H^{\text{low}}(z) = b_H^{\text{low}}(z), \quad z \in \Delta_1,$$
$$t_H^{\text{low}}(z) = \lambda_0 b_{\widetilde{H}}^{\text{up}}(z) + (1-\lambda_0) f^{\text{up}}(z), \quad z \in \Delta_0,$$
$$t_H^{\text{low}}(z) = \lambda_2 f^{\text{low}}(z) + (1-\lambda_2) f^{\text{up}}(z), \quad z \in \Delta_2,$$

with $\lambda_0 = (f^{\text{up}}(x_H^l) - b_H^{\text{up}}(x_H^l))/(f^{\text{up}}(x_H^l) - b_{\widetilde{H}}^{\text{up}}(x_H^l))$, guaranteeing the continuity of t_H^{low} at x_H^l. Similarly $\lambda_2 = (f^{\text{up}}(x_H^r) - b_H^{\text{up}}(x_H^r))/(f^{\text{up}}(x_H^r) - f^{\text{low}}(x_H^r))$.

The second selection t_H^{up} coincides with b_H^{up} on Δ_1 and with t_H^{low} on $\Delta_0 \cup \Delta_2$. Thus $t_H^{\text{low}}, t_H^{\text{up}}$ partition Graph(F) into two subgraphs (as depicted in Fig. 5.1), such that

$$F(x) = F_1(x) \bigcup F_2(x),$$

with $f_1^{\text{low}}(x) = f^{\text{low}}(x)$, $f_1^{\text{up}}(x) = t_H^{\text{low}}(x)$, $f_2^{\text{low}}(x) = t_H^{\text{up}}(x)$, $f_2^{\text{up}}(x) = f^{\text{up}}(x)$ the lower and upper boundaries of $F_1(x)$ and $F_2(x)$ respectively.

Note that F_2 is segmental, but F_1 still has non-convex images. The graph of F_1 has a unique hole $\widetilde{H}$, while H is not a hole of F_1 or F_2. Next, using the same technique, the hole $\widetilde{H}$ can be eliminated by subdividing F_1 into two segment functions. The union of these two multifunctions with F_2 gives a TMSR of the original SVF. It is easy to verify that in this example the same TMSR is obtained when we first eliminate $\widetilde{H}$ and then H.

In the following we describe the construction of a TMSR of $F \in \mathcal{F}([a,b])$ in the form of an algorithm. We use here the notation $t^{\text{up}}(F)$ ($t^{\text{low}}(F)$) for f^{up} (f^{low}) of F.

Construction. Given a multifunction $F \in \mathcal{F}([a,b])$

(i) Set $i = 1$, $F_i = F$, $I = 1$.
(ii) **While** $\langle i \leq I \rangle$

If $\langle |\mathcal{H}(F_i)| > 0\rangle$
(1) Choose any hole $H \in \mathcal{H}(F_i)$.
(2) Construct $t_H^{\text{low}}, t_H^{\text{up}}$ of F_i.
(3) $I = I + 1$.
(4) $F_I(x) = F_i(x) \bigcap [t^{\text{low}}(F_i)(x), t_H^{\text{low}}(x)], x \in [a, b]$.
(5) $F_i(x) = F_i(x) \bigcap [t_H^{\text{up}}(x), t^{\text{up}}(F_i)(x)], x \in [a, b]$.
Else $i = i + 1$.
End While

(iii) For any $x \in [a, b]$, $F(x) = \bigcup_{i=1}^{I} F_i(x)$.

The obtained segment functions $\{F_i\}$ are not ordered in the sense of Definition 4.1. This can be corrected by renumbering these multifunctions.

At this point the construction yields a MS-representation with significant selections as MS-boundaries. Lemma 5.6 shows that this MSR is topological. But first we note the following observation

Remark 5.5. For any two functions f_1, f_2 defined on Δ, let g_1, g_2 be $g_1 = \lambda f_1 + (1-\lambda) f_2$, $g_2 = \widetilde{\lambda} f_1 + (1-\widetilde{\lambda}) f_2$, $\lambda, \widetilde{\lambda} \in [0, 1]$. Clearly, any function $h = \mu g_1 + (1-\mu) g_2$, $\mu \in [0, 1]$ can be presented on Δ as a convex combination of f_1 and f_2.

Lemma 5.6. *Let $G(x) \subset F(x)$ for $x \in [a, b]$ such that $\mathcal{H}(G) \subset \mathcal{H}(F)$ and $t^{\text{up}}(G)$, $t^{\text{low}}(G)$ are topological selections of F. Then any topological selection of G is a topological selection of F.*

Proof. Clearly by the assumptions of the lemma, χ_G is a subset of χ_F. Let s be a topological selection of G. By Definitions 4.5 and 5.1, if s either coincides with $t^{\text{up}}(G)$ or $t^{\text{low}}(G)$ then by assumption it is a topological selection of F. If $s = t_H^{\text{up}}$ or t_H^{low} for some $H \in \mathcal{H}(G)$, then on any interval $\Delta \neq \Delta_H$ specified by χ_G

$$s|_\Delta = \lambda_n g_n^{\text{low}} + (1-\lambda_n) g_n^{\text{up}}, \tag{16}$$

where g_n^{low} and g_n^{up} are determined by the piecewise natural MS-representation of G, i.e. $g_n^{\text{low}}, g_n^{\text{up}} \in \partial G = \{b_H^{\text{low}}, b_H^{\text{up}} : H \in \mathcal{H}(G)\} \bigcup \{t^{\text{low}}(G), t^{\text{up}}(G)\}$. But since $\mathcal{H}(G) \subset \mathcal{H}(F)$ and in view of the assumption on $t^{\text{low}}(G), t^{\text{up}}(G)$ and Remark 5.5, s is a topological selection of F on each interval Δ determined by χ_F, and therefore on $[a, b]$. $\square$

The advantage of a TMSR is that its boundaries inherit the continuity properties of F. This follows from Theorem 3.7 and Definition 5.1. Thus we have

Corollary 5.7. *Let $F \in \mathcal{F}([a, b])$ and let $\mathcal{R}$ be a TMSR of F. Then for any $f \in \mathcal{B}(\mathcal{R}, F)$*

$$\omega_{[a,b]}(f, \delta) \leq \omega_{[a,b]}(v_F, \delta), \qquad \delta > 0. \tag{17}$$

Moreover if $F \in \text{Lip}([a, b], L)$ then $f \in \text{Lip}([a, b], L)$.

5.2. Conditions for uniqueness

In general a TMSR is not unique. Fig. 5.2(a) illustrates the graph of a multifunction F with two possible TMSRs of the form $F = F_1 \bigcup F_2 \bigcup F_3$. Here F_2 is not defined uniquely on Δ_{H_1}, as is demonstrated in Fig. 5.2(b), (c).

Moreover, there exist other TMSRs in which F_2 is replaced on $\Delta_{H_2} = \Delta_{H_3}$ by a union of several segment functions, since on this interval the topological selections $t_{H_1}^{\text{low}}, t_{H_1}^{\text{up}}$ are not uniquely defined.

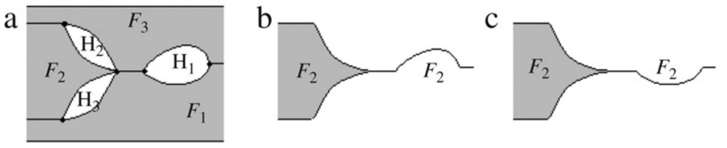

Fig. 5.2. (a) — The graph of F with two TMSRs. (b), (c) — The graphs of two possible F_2.

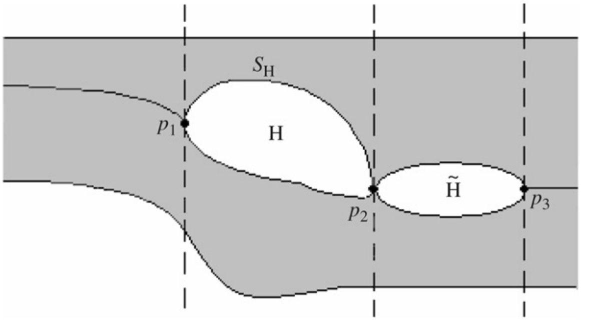

Fig. 5.3. F with non-unique pairs of topological selections.

The main source of non-uniqueness in our construction of a TMSR is the non-uniqueness in the definition of a pair of topological selections of a hole. The pair t_H^{low}, t_H^{up} for $H \in \mathcal{H}(F)$ is unique if on $[a, b] \setminus \Delta_H$, the graph of $t_H^{\text{low}} = t_H^{\text{up}}$ is contained in $int(\text{Graph}(F))$. Non-uniqueness can occur when a pair of topological selections $t_{\underset{\sim}{H}}^{\text{low}}$, t_H^{up} of a hole H passes through a point of change of topology associated with another hole $\widetilde{H}$, as in the examples in Figs. 5.2 and 5.3.

To eliminate this source of non-uniqueness we use in the construction another type of pairs of topological selections.

Definition 5.8. For each $H \in \mathcal{H}(F)$ we define a **special topological pair** of topological selections $(\theta_H^{\text{up}}, \theta_H^{\text{low}})$ by:

$$\theta_H^{\text{up}} = b_H^{\text{up}}, \qquad \theta_H^{\text{low}} = b_H^{\text{low}} \quad \text{on } \Delta_H.$$

On every $\Delta_i \neq \Delta_H$ determined by χ_F $\theta_H^{\text{up}} = \theta_H^{\text{low}}$ except if the graphs of θ_H^{up} and θ_H^{low} pass through a PCT associated with $\widetilde{H} \neq H$. Then on $\Delta_{\widetilde{H}}$, $\theta_H^{\text{up}} = b_{\widetilde{H}}^{\text{up}}$ and $\theta_H^{\text{low}} = b_{\widetilde{H}}^{\text{low}}$.

Remark 5.9. Note that the pair $(\theta_H^{\text{up}}, \theta_H^{\text{low}})$ coincides with the pair $(t_H^{\text{up}}, t_H^{\text{low}})$ whenever $(t_H^{\text{up}}, t_H^{\text{low}})$ is uniquely defined, otherwise it is associated with more than one hole (as in Fig. 5.3).

Still, in some cases there is ambiguity in the construction of $(\theta_H^{\text{up}}, \theta_H^{\text{low}})$ (see Fig. 5.2). The uniqueness of the special topological pairs can be guaranteed if all the points of change of topology of F are regular in the sense of the following definition.

Definition 5.10. In the notation of Remark 4.3, a PCT (x, y) is called **regular** if for i and n such that $x \in \Delta_i$, $y \in F_n^i(x)$,

$$\lim_{z \to x} \mu(F_n^i(z)) > 0.$$

It is easy to see that in Fig. 5.2 $\lim \mu(F_2(z)) \to 0$ as $z \to x^r_{H_3}$. Thus the PCT at $x^r_{H_3} = x^r_{H_2}$ is not regular, and $(\theta^{\mathrm{up}}_{H_1}, \theta^{\mathrm{low}}_{H_1})$ is not unique. On the other hand in Fig. 5.3 the PCTs p_1, p_2, p_3 are regular and F has only one special topological pair. The pair $(\theta^{\mathrm{up}}_{H}, \theta^{\mathrm{low}}_{H})$ coincides with the pair $(\theta^{\mathrm{up}}_{\widetilde{H}}, \theta^{\mathrm{low}}_{\widetilde{H}})$. The TMSR $F_1 \bigcup F_2$ with $F_1(x) = [f^{\mathrm{low}}(x), \theta^{\mathrm{low}}_{H}]$, $F_2(x) = [\theta^{\mathrm{up}}_{H}, f^{\mathrm{up}}(x)]$ is minimal in the sense that there is no function among the segment functions that can be removed from the representation.

In general we have

Lemma 5.11. *For F with only regular PCTs, each $H \in \mathcal{H}(F)$ determines a unique special topological pair $(\theta^{\mathrm{up}}_{H}, \theta^{\mathrm{low}}_{H})$.*

Proof. It is easy to see that any regular point of change of topology (x, y) satisfies $y \neq f^{\mathrm{low}}(x), f^{\mathrm{up}}(x)$, and is associated with one or two holes. In the latter case $\Delta_{\widetilde{H}} \bigcap \Delta_H = \{x\}$, where $H, \widetilde{H}$ are the two holes. (See Fig. 5.3).

To see that $(\theta^{\mathrm{up}}_{H}, \theta^{\mathrm{low}}_{H})$ is unique for $H \in \mathcal{H}(F)$, it is sufficient to consider θ^{up}_{H}. The proof is by induction. On Δ_H $\theta^{\mathrm{up}}_{H} = b^{\mathrm{up}}_{H}$. Suppose that θ^{up}_{H} is defined uniquely on $[x_j, x_{j+k}] \supset \Delta_H$, $x_j, x_{j+k} \in \chi_F$. For $j > 0$ if $(x_j, \theta^{\mathrm{up}}_{H}(x_j))$ is a regular PCT of some $\widetilde{H} \in \mathcal{H}(F)$ and $x_j = x^r_{\widetilde{H}}$ then by definition $\theta^{\mathrm{up}}_{H} = b^{\mathrm{up}}_{\widetilde{H}}$ on $\Delta_{\widetilde{H}}$. Otherwise θ^{up}_{H} is determined uniquely on $[x_{j-1}, x_j]$ by Definition 5.1. Thus θ^{up}_{H} is defined uniquely on $[x_{j-1}, x_{j+k}]$. The proof for x_{j+k} with $j+k < M$ is similar. $\square$

Finally, we conclude from the last lemma, Remark 5.9 and Definition 5.1 that

Corollary 5.12. *Consider F with only regular PCTs, and let $(\theta^{\mathrm{up}}_{H_i}, \theta^{\mathrm{low}}_{H_i})$ $i = 1, \ldots, m$ be its distinct special topological pairs. Then for $i \neq j$ there exists $\varepsilon \in \{-1, 1\}$ such that*

$$\varepsilon\theta^{\mathrm{low}}_{H_i} < \varepsilon\theta^{\mathrm{low}}_{H_j}, \quad \text{and} \quad \varepsilon\theta^{\mathrm{up}}_{H_i} < \varepsilon\theta^{\mathrm{up}}_{H_j}, \quad x \in [a, b].$$

Corollary 5.12 leads to the uniqueness result,

Theorem 5.13. *If all PCTs of F are regular, then there is a unique TMSR with minimal number of segment functions. This TMSR is determined by the distinct special topological pairs among*

$$\{(\theta^{\mathrm{up}}_{H}, \theta^{\mathrm{low}}_{H}) : H \in \mathcal{H}(F)\} \tag{18}$$

ordered according to (13).

Proof. By Lemma 5.11 the set (18) is uniquely determined. Now by Corollary 5.12 all distinct pairs among (18) can be ordered, such that,

$$f^{\mathrm{low}} \le \theta^{\mathrm{low}}_{H_1} \le \theta^{\mathrm{up}}_{H_1} \le \theta^{\mathrm{low}}_{H_2} \le \cdots \le \theta^{\mathrm{low}}_{H_m} \le \theta^{\mathrm{up}}_{H_m} \le f^{\mathrm{up}}.$$

Let $F_1 = [f^{\mathrm{low}}, \theta^{\mathrm{up}}_{H_1}]$, $F_{m+1} = [\theta^{\mathrm{up}}_{H_m}, f^{\mathrm{up}}]$ and $F_i = [\theta^{\mathrm{up}}_{H_{i-1}}, \theta^{\mathrm{low}}_{H_i}]$, $i = 2, \ldots, m$. We now show that $F = \bigcup_{i=1}^{m+1} F_i$. Observe that for a given $x \in [a, b]$, $(\bigcup_{i=1}^{m+1} F_i)(x) = [f^{\mathrm{low}}(x), \beta_1] \bigcup [\alpha_{m+1}, f^{\mathrm{up}}(x)] \bigcup_{i=2}^{m} [\alpha_i, \beta_i]$, with $\alpha_i = \theta^{\mathrm{up}}_{H_{i-1}}(x)$, $\beta_i = \theta^{\mathrm{low}}_{H_i}(x)$. Now, for $x \in (x^l_{H_i}, x^r_{H_i})$, $\beta_i < \alpha_{i+1}$ and $(\beta_i, \alpha_{i+1}) \subset H_i(x)$, and therefore

$$\left(\bigcup_{i=1}^{m+1} F_i\right)(x) = [f^{\mathrm{low}}(x), f^{\mathrm{up}}(x)] \setminus \bigcup_{\{H \in \mathcal{H}(F) : x \in \Delta_H\}} (b^{\mathrm{low}}_{H}(x), b^{\mathrm{up}}_{H}(x)).$$

Since $F(x)$ coincides with the right-hand side of the above equality, $F_1, \ldots, F_{m+1}$ determine a TMSR.

It remains to show that any other TMSR consists of more than $m+1$ segment functions. Recall that by Lemma 4.4 in any TMSR, the MS-boundaries contain for each $H \in \mathcal{H}(F)$ a pair of topological selections corresponding to H, namely one coinciding on Δ_H with b_H^{up} and the other with b_H^{low}. Since by definition each pair $(\theta_H^{\text{up}}, \theta_H^{\text{low}})$ is associated with the maximal possible number of holes, any other TMSR consists of more segment functions, than the TMSR considered in the theorem. □

6. Positive linear operators on SVFs

In this section we extend the definition of a class of positive operators acting on real-valued functions $f : [a, b] \to \mathbb{R}$ to multifunctions in $\mathcal{F}([a, b])$, based on a given MS-representation. We measure the quality of the approximation in the Hausdorff metric.

In the following P is a positive linear operator defined on continuous real-valued functions. In the case of a sample-based positive linear operator of the form

$$(P_n f)(x) = \sum_{i=1}^{n} \alpha_i(x) f(x_i), \quad a \le x_1 \le \cdots \le x_n \le b, \quad \alpha_i(x) \ge 0, i = 0, \ldots, n, \quad (19)$$

the frequently used extension of P to multifunctions is based on Minkowski sum of sets,

$$(P_n F)(x) = \sum_{i=1}^{n} \alpha_i(x) F(x_i), \quad x \in [a, b].$$

For segment functions such a definition is equivalent to $P_n F = [P_n f^{\text{low}}, P_n f^{\text{up}}]$.

Thus for a segment function F we define PF as

$$(PF)(x) = [(Pf^{\text{low}})(x), (Pf^{\text{up}})(x)], \quad x \in [a, b].$$

Note that PF is a well-defined segment function, since P is a positive operator. Clearly for a continuous segment function F, PF approximates F whenever P approximates continuous real-valued functions. From now on we consider only approximating positive operators.

Definition 6.1. Let $F \in \mathcal{F}([a, b])$ with a given MS-representation $\mathcal{R}$. We define

$$(P_{\mathcal{R}} F)(x) = \bigcup_{n=1}^{N} (PF_n)(x) = \bigcup_{n=1}^{N} [(Pf_n^{\text{low}})(x), (Pf_n^{\text{up}})(x)], \quad x \in [a, b],$$

where $\mathcal{B}(\mathcal{R}, F) = \{f_n^{\text{low}}, f_n^{\text{up}}, n = 1, \ldots, N\}$.

It is easy to conclude from this definition that $P_{\mathcal{R}} F$ approximates F whenever $\mathcal{R}$ is a continuous MS-representation.

Theorem 6.2. *Let P be a positive linear approximation operator. Then for $F \in \mathcal{F}([a, b])$ with MS-representation $\mathcal{R}$*

$$\text{haus}(P_{\mathcal{R}} F(x), F(x)) \le \max_{f \in \mathcal{B}(\mathcal{R}, F)} |Pf(x) - f(x)|. \quad (20)$$

Proof. Let $\mathcal{R} = \{F_n, n = 1, \ldots, N\}$ be the given MSR of F. For any $y \in F(x)$, there exists $1 \le i \le N$ such that $y \in F_i(x)$. Then by the definition of Hausdorff distance

$$\operatorname{dist}(y, (P_{\mathcal{R}}F)(x)) \le \operatorname{dist}(y, (P_{\mathcal{R}}F_i)(x)) \le \operatorname{haus}(F_i(x), (P_{\mathcal{R}}F_i)(x)).$$

Since F_i, $P_{\mathcal{R}}F_i$ are segment functions, we get in view of (1)

$$\begin{aligned}\operatorname{dist}(y, (P_{\mathcal{R}}F)(x)) &\le \max\{|f_i^{\mathrm{up}}(x) - Pf_i^{\mathrm{up}}(x)|, |f_i^{\mathrm{low}}(x) - Pf_i^{\mathrm{low}}(x)|\} \\ &\le \max_{f \in \mathcal{B}(\mathcal{R},F)} |f(x) - Pf(x)|.\end{aligned}$$

Thus

$$\sup_{y \in F(x)} \operatorname{dist}(y, (P_{\mathcal{R}}F)(x)) \le \max_{f \in \mathcal{B}(\mathcal{R},F)} |f(x) - Pf(x)|.$$

Similarly

$$\sup_{p \in (P_{\mathcal{R}}F)(x)} \operatorname{dist}(p, F(x)) \le \max_{f \in \mathcal{B}(\mathcal{R},F)} |f(x) - Pf(x)|. \quad \square$$

The regularity (modulus of continuity) of f determines bounds on $|f(x) - Pf(x)|$ [6]. As is shown in Section 4, every continuous MSR contains all boundaries of F as parts of its MS-boundaries. Thus a good MS-representation should have MS-boundaries with regularity not worse than the regularity of the boundaries of F. Our topological representation meets this condition. We use it below to define the operation of P on F.

Definition 6.3. For $F \in \mathcal{F}([a, b])$ we define $PF = P_{\mathcal{R}^*}F$, with $\mathcal{R}^*$ a TMSR of F. In case F has only regular PCTs this definition is unique.

With this definition it is possible to obtain from Theorem 6.2 approximation estimates in terms of the regularity of F as a multifunction and not in terms of the regularity of its boundaries.

Theorem 6.4. *Let P^δ be a positive linear approximation operator satisfying for any $x \in [a, b]$*

$$|(P^\delta f)(x) - f(x)| \le C\, \omega_{[a,b]}(f, \phi(x, \delta)), \qquad f \in \mathcal{C}([a, b]),$$

with δ a small parameter and $\phi : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function, non-decreasing in its second argument such that $\phi(x, 0) = 0$.

Then for $F \in \mathcal{F}([a, b])$ with a TMSR $\mathcal{R}^*$

$$\operatorname{haus}((P^\delta_{\mathcal{R}^*}F)(x), F(x)) \le C\omega_{[a,b]}(v_F, \phi(x, \delta)), \qquad x \in [a, b]. \tag{21}$$

In particular, if $F \in \mathrm{Lip}([a, b], L)$ then $\operatorname{haus}((P^\delta_{\mathcal{R}^*}F)(x), F(x)) \le CL\phi(x, \delta)$, $x \in [a, b]$.

Proof. The proof follows from Theorem 6.2 and Corollary 5.7. $\square$

7. Examples of positive linear operators

In this section we consider the analogues of the Bernstein polynomial operators and the Schoenberg spline operators for SVFs in $\mathcal{F}([a, b])$ and derive error estimates. We illustrate our set-valued extension of these operators with two examples.

7.1. Bernstein operators

For a real-valued function $f \in \mathcal{C}([0,1])$ the Bernstein operator B_m is defined as

$$(B_m f)(x) = \sum_{i=0}^{m} \binom{m}{i} x^i (1-x)^{m-i} f\left(\frac{i}{m}\right). \tag{22}$$

The known error estimate (see [6, Chapter 10]) is

$$|f(x) - (B_m f)(x)| \le C\omega_{[0,1]}(f, \sqrt{x(1-x)/m}), \tag{23}$$

where C does not depend on f or m.

Now for $F \in \mathcal{F}([0,1])$ we define the Bernstein operator as

$$(B_m F)(x) = \bigcup_{n=1}^{N} [(B_m f_n^{\mathrm{low}})(x), (B_m f_n^{\mathrm{up}})(x)], \quad x \in [0,1], \tag{24}$$

with $f_n^{\mathrm{low}}, f_n^{\mathrm{up}}, n = 1, \dots, N$ topological selections defining a TMSR of F.

Remark 7.1. It is easy to conclude that for m large enough the representation (24) of $B_m F$ is a natural MS-representation.

Application of Theorem 6.4 yields

Corollary 7.2. *Let* $F \in \mathcal{F}([0,1])$. *Then*

$$\mathrm{haus}((B_m F)(x), F(x)) \le C\omega_{[0,1]}(v_F, \sqrt{x(1-x)/m}), \quad x \in [0,1].$$

Moreover for $F \in \mathrm{Lip}([0,1], L)$

$$\mathrm{haus}((B_m F)(x), F(x)) \le CL(\sqrt{x(1-x)/m}), \quad x \in [0,1].$$

Here C is a generic constant independent of F.

To illustrate our approach we present the following example. Consider

$$F(x) = \begin{cases} \left[-\frac{1}{2}, \frac{1}{2}\right], & x \in [0,1] \setminus \left[\frac{1}{4}, \frac{3}{4}\right] \\ \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \left[-\sqrt{\frac{1}{4}^2 - \left(x - \frac{1}{2}\right)^2}, \sqrt{\frac{1}{4}^2 - \left(x - \frac{1}{2}\right)^2}\right], & x \in \left[\frac{1}{4}, \frac{3}{4}\right]. \end{cases} \tag{25}$$

This multifunction is depicted in gray in (a), (b), (c), (d) of Fig. 7.1. The MS-boundaries of the unique topological MS-representation of F and the two PCTs of F are shown in black in (a). The cross-sections of the approximating SVFs $B_9 F$, $B_{31} F$ and $B_{50} F$, are colored by black in (b), (c) and (d) respectively. The maximal error is attained at $x_1 = 0.25$ and $x_2 = 0.75$, which are the abscissas of the PCTs of F.

It can be shown that the MS-boundaries $\theta_H^{\mathrm{up}}, \theta_H^{\mathrm{low}}$ are Hölder continuous with exponent $1/2$ at $x = x_i$, $i = 1, 2$. Thus using Theorem 6.2 and (23) we get

$$\mathrm{haus}((B_m F)(x_i), F(x_i)) \le C\sqrt[4]{x_i(1-x_i)/m}, \quad i = 1, 2,$$

which explains the slow decay of the error at x_1, x_2.

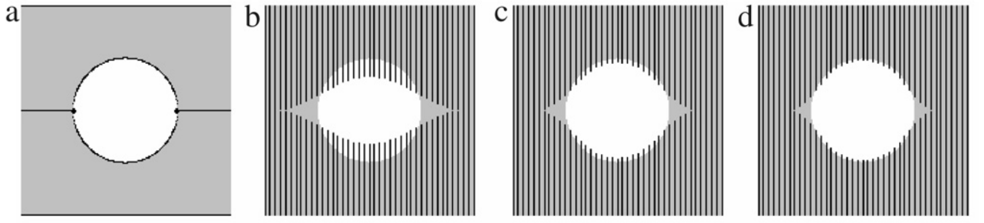

Fig. 7.1. (a) F — in gray. Four MS-boundaries of the TMSR — in black. Two PCTs of F — in black. (b), (c), (d) F — in gray. 40 cross-sections of B_9F, $B_{31}F$ and $B_{50}F$ respectively — in black.

7.2. Schoenberg operators

For a uniform partition $\chi = (x_0, \dots, x_k)$, $x_i = a + ih$, $h = (b-a)/k$, the Schoenberg spline operator for $f \in \mathcal{C}([a,b])$ is

$$(S_{m,h}f)(x) = \sum_{i=0}^{k} f(x_i) b_m(x/h - i), \tag{26}$$

where $b_m(x)$ is the B-spline of order m (degree $m-1$) with integer knots and support $[0, m]$. It is known (see [7, Chapter XII]), that

$$|(S_{m,h}f)(x) - f(x)| \le \left\lfloor \frac{m+1}{2} \right\rfloor \omega_{[a,b]}(f,h), \quad x \in [a + (m-1)h, b],$$

with $\lfloor x \rfloor$ the maximal integer not greater than x.

Remark 7.3. A better approximation result can be obtained using the symmetric Schoenberg operator, when $b_m(\cdot)$ in (26) is replaced by the centered B-spline $b_m(\cdot - m/2)$.

For $F \in \mathcal{F}([a,b])$ the set-valued analogue of the Schoenberg spline operator using a topological MS-representation with MS-boundaries $f_n^{\text{low}}, f_n^{\text{up}} : n = 1, \dots, N$ is defined by

$$(S_{m,h}F)(x) = \bigcup_{n=1}^{N} [(S_{m,h}f_n^{\text{low}})(x), (S_{m,h}f_n^{\text{up}})(x)], \quad x \in [a,b]. \tag{27}$$

Then by Theorem 6.4 we have

Corollary 7.4. (i) *For* $F \in \mathcal{F}([0,1])$

$$\text{haus}((S_{m,h}F)(x), F(x)) \le \lfloor (m+1)/2 \rfloor\, \omega_{[0,1]}(v_F, h), \quad x \in [a + (m-1)h, b].$$

(ii) *For* $F \in \text{Lip}([0,1], L)$

$$\text{haus}((S_{m,h}F)(x), F(x)) \le \lfloor (m+1)/2 \rfloor\, Lh, \quad x \in [a + (m-1)h, b].$$

Fig. 7.2 illustrates an approximation of F given in (25) by the Schoenberg spline operators based on the TMSR of F.

As in the case of Bernstein operators the maximal error is attained at the abscissas of the PCTs of F. By arguments similar to those in the case of Bernstein operators, we get that the error decays as $O(h^{1/2})$, which is much faster than the decay $O(h^{1/4})$ $(h = 1/m)$ in Fig. 7.1.

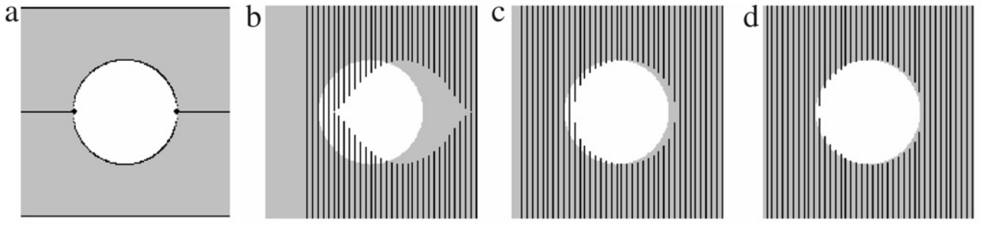

Fig. 7.2. (a) F — in gray. Four MS-boundaries of the TMSR — in black. Two PCTs of F — in black. (b), (c), (d) F — in gray. 40 cross-sections of $S_{3,0.1}F$, $S_{3,0.025}F$ and $S_{3,0.01}F$ respectively — in black.

Appendix A

Here we prove Theorem 3.6, namely that the boundaries of continuous SVFs of bounded variation are continuous. This follows from a basic property of CBV multifunctions, which we state in more generality than needed here. The property is the existence of a continuous selection through any point of the graph of a CBV multifunction. It is an extension of a result by Hermes [8] on the existence of a continuous selection of a CBV multifunction, and the proof is based on our construction of metric chains in [5]. A similar construction is used by Chistyakov [9] to prove a similar result but with CBV selections. Yet our result provides, in addition, an estimate of the modulus of continuity of the constructed selections.

Theorem A.1. *Let $F : [a,b] \to K(\mathbb{R}^n)$ be a CBV multifunction. Then through any point in* Graph(F) *there exists a continuous selection s satisfying*

$$\omega_{[a,b]}(s,\delta) \leq 4\,\omega_{[a,b]}(v_F,\delta), \quad \delta > 0. \tag{28}$$

In particular for $F \in$ Lip$([a,b],L)$*, $s \in$* Lip$([a,b],L)$.

Proof. For a fixed $(\widetilde{x},\widetilde{y}) \in \text{Graph}(F)$, $\widetilde{x} \in [a,b]$, $\widetilde{y} \in \mathbb{R}^n$ we construct "chains" as in [5]. Let $x_i = a + ih$, $i = 0,\dots,N$, $h > 0$, $Nh = b - a$ and let j be such that $x_j \leq \widetilde{x} \leq x_{j+1}$. Choose $y_j \in \Pi_{F(x_j)}(\widetilde{y})$, $y_k \in \Pi_{F(x_k)}(y_{k+1})$, $k = j-1, j-2, \dots, 0$, and similarly $y_{j+1} \in \Pi_{F(x_{j+1})}(\widetilde{y})$, $y_k \in \Pi_{F(x_k)}(y_{k-1})$, $k = j+2,\dots,N$.

As in [5, Section 4], we define the partition $\chi_N = \{x_0,\dots,x_j,\widetilde{x},x_{j+1},\dots,x_N\}$, a metric chain $\varphi = \{y_0,\dots,y_j,\widetilde{y},y_{j+1},\dots,y_N\}$ and the piecewise linear function $s_N(\chi,\varphi)$ interpolating the points $(\widetilde{x},\widetilde{y})$, (x_i,y_i), $i = 0,\dots,N$. Lemma 4.7 of [5] shows

$$\omega_{[a,b]}(s_N(\chi,\varphi),\delta) \leq 4\omega_{[a,b]}(v_F,\delta), \quad \delta > 0. \tag{29}$$

Also, by Corollary 4.4 of [5] for $F \in \text{Lip}([a,b],L)$

$$s_N(\chi,\varphi) \in \text{Lip}([a,b],L). \tag{30}$$

Observe, that the set of all such piecewise linear functions $\{s_N(\chi,\varphi) : N \in \mathbb{Z}_+\}$ is equicontinuous by (29) or (30) and is equibounded since the set $\{F(x) : a \leq x \leq b\}$ is bounded. Then by Arzelà-Ascoli theorem there exists a subsequence which converges to a continuous function s through $(\widetilde{x},\widetilde{y})$, satisfying (28). It is easy to see that s is a selection of F, since Graph(F) is closed. $\square$

Using the above theorem we prove that all the cross-sections of the holes of a CBV multifunction with images in $\mathbb{R}$ are convex.

Lemma A.2. *Let $F \in CBV([a,b])$, and let $H \in \mathcal{H}(F)$. Then for any $x \in (x_H^l, x_H^r)$ the cross-section $H(x) = \{y : (x,y) \in H\}$ is convex (an open interval).*

Proof. Assume the opposite, i.e. there exists $\widetilde{x} \in (x_H^l, x_H^r)$ such that $H(\widetilde{x})$ is not convex. Then there exists a segment $[y_1, y_2] \in co(H(\widetilde{x})) \setminus H(\widetilde{x}) \subset F(\widetilde{x})$ with $y_1 \le y_2$ and $(\widetilde{x}, y_1), (\widetilde{x}, y_2) \in \widetilde{\partial} H$. Let $\varepsilon > 0$ be such that $(\widetilde{x}, y_1 - \varepsilon), (\widetilde{x}, y_2 + \varepsilon) \in H$. By the connectivity of H there is a continuous path φ in H from $(\widetilde{x}, y_1 - \varepsilon)$ to $(\widetilde{x}, y_2 + \varepsilon)$.

Now, by Theorem A.1 there exists a continuous selection s of F through $(\widetilde{x}, y_1)$ which must intersect the path φ. Since $\varphi \in H$ and $s(x) \in F(x)$ for $a \le x \le b$, we get a contradiction. □

Corollary A.3. *For $F \in CBV([a,b])$, $b_H^{\text{low}}(x) < b_H^{\text{up}}(x)$, for $x \in (x_H^l, x_H^r)$. Moreover, for $x \in (x_H^l, x_H^r)$, $(x, b_H^{\text{low}}(x))$ and $(x, b_H^{\text{up}}(x))$ are the only points of $cl(H(x)) \setminus H(x)$.*

Next we prove the continuity of b_H^{low} and b_H^{up}. For that we need the following lemma.

Lemma A.4. *Let s be a continuous selection of $F \in CBV([a,b])$ and let $H \in \mathcal{H}(F)$. Then either $s(x) \ge b_H^{\text{up}}(x)$ or $s(x) \le b_H^{\text{low}}(x)$, $\forall x \in (x_H^l, x_H^r)$.*

Proof. We prove the first inequality. For $x \in (x_H^l, x_H^r)$, denote $y = s(x)$, $b_x = b_H^{\text{up}}(x)$ and let $s(x) \ge b_x$. Assume that for some $z \in (x_H^l, x_H^r)$ $s(z) \le b_H^{\text{low}}(z) = b_z$. By the connectivity of H there exists a continuous path $\varphi \in H$ from $(x, b_x - \varepsilon)$ to $(z, b_z + \varepsilon)$ for any small $\varepsilon > 0$. Since s is continuous and passes through the points $(x, s(x))$, $(z, s(z))$, with $s(x) > b_x - \varepsilon$, $s(z) < b_z + \varepsilon$, it must intersect $\varphi \in H$. This contradicts the definition of a selection. □

Lemma A.5. *For $F \in CBV([a,b])$ and $H \in \mathcal{H}(F)$, b_H^{up} and b_H^{low} are continuous on (x_H^l, x_H^r).*

Proof. We prove the lemma for b_H^{up}. Let $\widetilde{x} \in (x_H^l, x_H^r)$ and let $\{x_n\}$ be a sequence of points in (x_H^l, x_H^r) with $\lim_{n\to\infty} x_n = \widetilde{x}$. Consider the accumulation points of $\{b_H^{\text{up}}(x_n)\}$. By the continuity of F these points are in $cl(H(\widetilde{x}))$. Corollary A.3 guarantees that

$$cl(H(\widetilde{x})) \setminus H(\widetilde{x}) = \{b_H^{\text{low}}(\widetilde{x}), b_H^{\text{up}}(\widetilde{x})\} \subset F(\widetilde{x}). \tag{31}$$

In the following we show that $\liminf_{n\to\infty} b_H^{\text{up}}(x_n) \ge b_H^{\text{up}}(\widetilde{x})$, which in view of (31) implies the claim of the lemma.

By Theorem A.1 through any point $(x_n, b_H^{\text{up}}(x_n))$ there exists a continuous selection s_n satisfying (28). Let $\{s_{n_k}(x_{n_k})\}$ be a subsequence of $\{s_n(x_n)\}$ which tends to $\liminf_{n\to\infty} b_H^{\text{up}}(x_n) = \liminf_{n\to\infty} s_n(x_n)$ by construction.

Consider $s_{n_k}(x_{n_k}) - b_H^{\text{up}}(\widetilde{x}) = [s_{n_k}(x_{n_k}) - s_{n_k}(\widetilde{x})] + [s_{n_k}(\widetilde{x}) - b_H^{\text{up}}(\widetilde{x})]$. By Theorem A.1 the set $\{s_{n_k}\}$ is equicontinuous, i.e. for $|x_{n_k} - \widetilde{x}| \le \delta$, $|s_{n_k}(x_{n_k}) - s_{n_k}(\widetilde{x})| \le 4\omega(v_F, \delta)$. Therefore $s_{n_k}(x_{n_k}) - s_{n_k}(\widetilde{x})$ tends to zero when $k \to \infty$. From Lemma A.4 it follows that $s_{n_k}(\widetilde{x}) - b_H^{\text{up}}(\widetilde{x})$ is non-negative. Both these facts lead to $\lim_{k\to\infty} s_{n_k}(x_{n_k}) = \liminf_{n\to\infty} b_H^{\text{up}}(x_n) \ge b_H^{\text{up}}(\widetilde{x})$. □

Lemma A.6. *For $F \in CBV([a,b])$ and $H \in \mathcal{H}(F)$, the limits $\lim_{x\to x_H^l} b_H^{\text{up}}(x)$, $\lim_{x\to x_H^r} b_H^{\text{up}}(x)$, $\lim_{x\to x_H^l} b_H^{\text{low}}(x)$, $\lim_{x\to x_H^r} b_H^{\text{low}}(x)$ exist.*

Proof. We prove the existence of $\lim_{x\to x_H^l} b_H^{\text{up}}(x)$. Assume to the contrary, that there are $\{x_k\} \to x_H^l$ and $\{z_k\} \to x_H^l$ such that $\{b_H^{\text{up}}(x_k)\} \to f_1$, $\{b_H^{\text{up}}(z_k)\} \to f_2$ with $f_1 \ne f_2$. For any $\lambda \in [0,1]$, let $f_\lambda = \lambda f_1 + (1-\lambda) f_2$. By the continuity of b_H^{up} on (x_H^l, x_H^r) there exists

$\{\xi_k\}$, $\xi_k \in co\{x_k, z_k\}$ with $b_H^{\mathrm{up}}(\xi_k) = \lambda b_H^{\mathrm{up}}(x_k) + (1-\lambda) b_H^{\mathrm{up}}(z_k)$. Clearly, $\{b_H^{\mathrm{up}}(\xi_k)\} \to f_\lambda$. Thus for any $\lambda \in [0,1]$ we have $(x_H^l, f_\lambda) \in \widetilde{\partial} H$ implying that $co\{(x_H^l, f_1), (x_H^l, f_2)\} \subset \widetilde{\partial} H$, which contradicts Remark 3.3(i). Thus $f_1 = f_2 = \lim_{x \to x_H^l} b_H^{\mathrm{up}}(x)$.

Similarly, it can be shown that $\lim_{x \to x_H^r} b_H^{\mathrm{up}}(x)$, $\lim_{x \to x_H^l} b_H^{\mathrm{low}}(x)$, $\lim_{x \to x_H^r} b_H^{\mathrm{low}}(x)$ exist. □

We denote $b_H^{\mathrm{up}}(x_H^l) = \lim_{x \to x_H^l} b_H^{\mathrm{up}}(x)$, and similarly for the three other limits. Note that $b_H^{\mathrm{up}}(x_H^l), b_H^{\mathrm{low}}(x_H^l) \in F(x_H^l)$ and $b_H^{\mathrm{up}}(x_H^r), b_H^{\mathrm{low}}(x_H^r) \in F(x_H^r)$.

Lemma A.7. *For a hole H of $F \in CBV([a,b])$*

$$b_H^{\mathrm{low}}(x_H^l) = b_H^{\mathrm{up}}(x_H^l) \text{ if } \quad x_H^l > a \quad \text{and} \quad b_H^{\mathrm{low}}(x_H^r) = b_H^{\mathrm{up}}(x_H^r) \text{ if } \quad x_H^r < b.$$

Proof. We prove the lemma for $x_H^l > a$. Assume that $b_H^{\mathrm{low}}(x_H^l) \neq b_H^{\mathrm{up}}(x_H^l)$. For any $\widetilde{y} \in (b_H^{\mathrm{low}}(x_H^l), b_H^{\mathrm{up}}(x_H^l))$, $\widetilde{y} \in co(F(x_H^l))$. Consider $(x_H^l, \widetilde{y})$. Since $x_H^l > a$, $(x_H^l, \widetilde{y}) \notin H$ by the definition of x_H^l. Also $(x_H^l, \widetilde{y})$ cannot belong to some other hole $\widetilde{H} \neq H$, $\widetilde{H} \in \mathcal{H}(F)$, otherwise $H \bigcap \widetilde{H} \neq \emptyset$ in contradiction to the definition of a hole. Thus $(x_H^l, \widetilde{y}) \in \mathrm{Graph}(F)$. But by Remark 3.3(i) the point $(x_H^l, \widetilde{y})$ is singular, in contradiction to the continuity of F. □

Theorem 3.6 follows from Lemmas A.5–A.7.

Appendix B

In this appendix we show that the regularity properties of $F \in CBV([a,b])$ are inherited by its boundaries.

First we investigate local continuity away from PCTs. For $H \in \mathcal{H}(F)$ and for $\varepsilon > 0$ small enough denote

$$\Delta_H^\varepsilon = [x_H^l + \varepsilon, x_H^r - \varepsilon].$$

Lemma B.1. *Let $F \in \mathcal{F}([a,b])$. For $H \in \mathcal{H}(F)$ and for a given small $\varepsilon > 0$ there exists $\delta = \delta_{H,\varepsilon}$ such that for any $x, z \in \Delta_H^\varepsilon$, satisfying $|x - z| \leq \delta$,*

$$\begin{aligned} &\max\{|b_H^{\mathrm{low}}(x) - b_H^{\mathrm{low}}(z)|, |b_H^{\mathrm{up}}(x) - b_H^{\mathrm{up}}(z)|\} \\ &\leq \omega_{\Delta_H^\varepsilon}(F, |x-z|) \leq \omega_{[a,b]}(F, |x-z|). \end{aligned} \tag{32}$$

Proof. To prove the claim of the lemma it is sufficient to show that there exists $\delta = \delta_{H,\varepsilon} > 0$ small enough such that for all $x, z \in \Delta_H^\varepsilon$, $|z - x| \leq \delta$

$$\max\{|b_H^{\mathrm{low}}(x) - b_H^{\mathrm{low}}(z)|, |b_H^{\mathrm{up}}(x) - b_H^{\mathrm{up}}(z)|\} \leq \mathrm{haus}(F(x), F(z)).$$

Assume to the contrary that there exist sequences $\{x_n\}, \{z_n\} \subset \Delta_H^\varepsilon$ with $\lim_{n \to \infty} |x_n - z_n| = 0$ such that for all n large enough

$$|f(x_n) - f(z_n)| > \mathrm{haus}(F(x_n), F(z_n)), \tag{33}$$

for f either b^{low} or b^{up}.

Let $\min_{x \in \Delta_H^\varepsilon} |b_H^{\mathrm{low}}(x) - b_H^{\mathrm{up}}(x)| = \gamma$. Clearly, $\gamma > 0$. By the uniform continuity of b_H^{low} and b_H^{up} on Δ_H^ε there exists $\delta = \delta_{H,\varepsilon} > 0$ such that for any $x, z \in \Delta_H^\varepsilon$ satisfying $|z - x| \leq \delta$

$$b_H^{\mathrm{up}}(z) - b_H^{\mathrm{low}}(x) \geq \gamma/2. \tag{34}$$

Now choose N such that $|x_n - z_n| \leq \delta$ for $n \geq N$. Consider the case $f = b_H^{\text{low}}$. Let $f(x_n) \geq f(z_n)$. We claim that

$$\Pi_{F(z_n)}(f(x_n)) = f(z_n). \tag{35}$$

If not, then by (34) $\Pi_{F(z_n)}(f(x_n)) \geq b_H^{\text{up}}(z_n) \geq f(x_n) + \gamma/2$, in contradiction to the continuity of F. The equality (35) implies that

$$|f(x_n) - f(z_n)| \leq \text{haus}(F(x_n), F(z_n)), \quad n > N$$

which contradicts (33). $\square$

We extend this local result to the global result,

Theorem B.2. *Let $F \in \mathcal{F}([a,b])$. Then for a given $H \in \mathcal{H}(F)$*

$$\max\{\omega_{\Delta_H}(b_H^{\text{low}}, |x-z|), \omega_{\Delta_H}(b_H^{\text{up}}, |x-z|)\} \leq \omega_{[a,b]}(v_F, |x-z|), \tag{36}$$

with v_F defined as in (3).

Proof. Let f be either b_H^{up} or b_H^{low}. By Lemma B.1 for any $\varepsilon > 0$ there exist $\delta_{H,\varepsilon} > 0$ and a partition $\chi_n = \{x_0, x_1, \ldots, x_n\}$ of Δ_H^ε with $x_H^l + \varepsilon = x_0 < x_1 < \cdots < x_n = x_H^r - \varepsilon$ and with $x_{i+1} - x_i < \delta_{H,\varepsilon}$, $i \in \{0, \ldots, n-1\}$, such that

$$|f(x) - f(z)| \leq \text{haus}(F(x), F(z)), \tag{37}$$

when x, z belong to the same interval $[x_i, x_{i+1}]$.

For $x, z \in \Delta_H^\varepsilon$ with $x \in [x_i, x_{i+1}]$, $z \in [x_j, x_{j+1}]$, $0 \leq i < j \leq n-1$ it follows from the triangle inequality and (37) that

$$\begin{aligned}|f(x) - f(z)| &\leq |f(x) - f(x_{i+1})| + \sum_{l=i+1}^{j-1} |f(x_{l+1}) - f(x_l)| + |f(z) - f(x_j)| \\ &\leq \text{haus}(F(x), F(x_{i+1})) + \sum_{l=i+1}^{j-1} \text{haus}(F(x_l), F(x_{l+1})) + \text{haus}(F(z), F(x_j)).\end{aligned}$$

Now using the definition of the variation of F and (3) we get for $x, z \in \Delta_H^\varepsilon$

$$|f(x) - f(z)| \leq V_x^z(F) = v_F(z) - v_F(x) \leq \omega_{[a,b]}(v_F, |x-z|).$$

Taking the supremum over $|x-z|$, $x, z \in \Delta_H^\varepsilon$ we obtain $\omega_{\Delta_H^\varepsilon}(f, |x-z|) \leq \omega_{[a,b]}(v_F, |x-z|)$ for any $\varepsilon > 0$. In view of Theorem 3.6, this leads to the claim of the theorem. $\square$

We have a stronger result in the special case of Lipschitz continuous SVFs.

Theorem B.3. *If $F \in \text{Lip}([a,b], L)$, then for any $H \in \mathcal{H}(F)$*

$$b_H^{\text{up}} \in \text{Lip}(\Delta_H, L) \quad \textit{and} \quad b_H^{\text{low}} \in \text{Lip}(\Delta_H, L). \tag{38}$$

Proof. Let f be either b_H^{up} or b_H^{low} and let $\chi_n = \{x_0, x_1, \ldots, x_n\}$ be a partition of Δ_H^ε as in the proof of Theorem B.2. Thus on any $[x_i, x_{i+1}]$, $i \in \{0, \ldots, n-1\}$ it follows from (37) that

$$|f(x) - f(z)| \leq \text{haus}(F(x), F(z)) \leq L|x-z|, \quad x, z \in [x_i, x_{i+1}] \subset \Delta_H^\varepsilon.$$

Now let $x \in [x_i, x_{i+1}]$ and $z \in [x_j, x_{j+1}]$, where $0 \le i < j \le n-1$. Thus

$$|f(x) - f(z)| \le |f(x) - f(x_{i+1})| + \sum_{l=i+1}^{j-1} |f(x_{l+1}) - f(x_l)| + |f(z) - f(x_j)|$$
$$\le L|x_{i+1} - x| + \sum_{l=i+1}^{j-1} L|x_{l+1} - x_l| + L|y - x_j| \le L|z - x|.$$

Since for any $\varepsilon > 0$ $|f(x) - f(z)| \le L|z - x|$ and in view of Theorem 3.6, the claim of the theorem follows. $\square$

The proof of Theorem 3.7 is obtained by combining Theorems B.2, B.3 and 3.5.

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